Evidence for super-exponentially accelerating atmospheric carbon dioxide growth

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Abstract

We analyze the growth rates of atmospheric carbon dioxide and human population, by comparing the relative merits of two benchmark models, the exponential law and the finite-time-singular (FTS) power law. The later results from positive feedbacks, either direct or mediated by other dynamical variables, as shown in our presentation of a simple endogenous macroeconomic dynamical growth model. Our empirical calibrations confirm that human population has decelerated from its previous super-exponential growth until 1960 to "just" an exponential growth, but with no sign of more deceleration. As for atmospheric CO2 content, we find that it is at least exponentially increasing and most likely characterized by an accelerating growth rate as off 2009, consistent with an unsustainable FTS power law regime announcing a drastic change of regime. The coexistence of a quasi-exponential growth of human population with a super-exponential growth of carbon dioxide content in the atmosphere is a diagnostic that, until now, improvements in carbon efficiency per unit of production worldwide has been dramatically insufficient.

1 Introduction

Today humanity uses the equivalent of 1.5 planets to provide the resources we use and absorb our waste. This means it now takes the Earth one year and six months to regenerate what we use in a year.[1] — Is humanity inevitably doomed?

During the 1960, leaders were most concerned about human population growth (see for instance [28]) and about depletion of energy resources (see for example the first report by the Club of Rome [15] and its recent reassesse-

[1] `http://www.footprintnetwork.org/en/index.php/GFN/page/world_footprint/`

ment [10]). As a matter of fact, the growth rate[2] of human population has peaked in the late 1960s and although population is still growing, it is no longer the prime concern of policy leaders. This may be ill-advised as we show below that population growth is not decelerating anymore, being on a stable exponential (proportional) growth trajectory.

More recently, scientist and politicians became aware of global warming (see [30] for a historic overview), perhaps due to or augmented by anthropogenic effects (we do not enter this debate). We focus here on the undisputed fact that, due to the massive use of fossil energies, the world economy emits, among many other products, large amounts of carbon dioxide into the atmosphere. Part of this carbon dioxide is later absorbed by the oceans and plants. The fraction of carbon dioxide found in the atmosphere is currently around 50% of the total anthropogenic emissions, with a slight upward trend [16]. Once in the atmosphere, this CO_2 is thought to play a pivotal role in global warming. In a recent Nature issue [17], climate change due to CO_2 emissions is identified as one of the most pressing problems that mankind needs to address.

Ref. [29] discuss the IPAT identity which identifies the most important factors which drive carbon dioxide emissions. They write carbon dioxide emissions as the result of three factors

$$I = P \cdot A' \cdot T, \tag{1}$$

where I (impact) denotes the carbon dioxide emissions, P is human population, A' represents the affluence (measured as gross world product per capita) and T is technology.

The IPAT identity is useful to help thinking about the contributions of different variables and has been extensively used and discussed in the literature (see for instance [3]). However, because one deals fundamentally with a complex dynamical system driven by entangled feedback loops with delays, the IPAT identity falls short, in our opinion, of providing the framework to understand the inter-relationships between the dynamical variables. It is especially important to develop a dynamical framework with delays, when studying the time-evolution of global variables such as atmospheric carbon dioxide content and human population. Therefore, motivated by a dynamical view of the human-Earth system, we present here a framework borrowing from the theory of endogenous macroeconomic growth [14, 18], whose feedback loops are shown to generate robust regimes of super-exponential

[2]The growth rate r of the human population (or of any other variable) is defined by expression (2). Thus, a constant growth rate corresponds to a population growing exponentially, with a doubling time given by $(\log 2)/r$. As the present growth rate is $r(2010) \approx 1.8\%$ per year, this gives a present doubling time of 38.5 years. If nothing changes, the present 6.8 billion people will be more than 13 billion in 2050! This is in contradiction with projections of OECD for instance and other international organizations, which optimistically expect human population to stabilize around 9 billion individuals.

growth. Mathematically, these regimes can be described by simple equations, whose solutions exhibit finite-time singular (FTS) power law behaviors. The interest in such solutions is that they point to change of regimes [7,12,19].

The article is organized as follows. We presents a simple mathematical framework to model growth, first for a single variable like population in the presence of positive feedback, and then with several coupled variables, such as population, capital and technology. Two benchmark models, the exponential law and the FTS power law, are obtained as limiting cases of the theoretical framework. Then we describe the results of the calibration of these two models to some of the most extensive data sources on human population and atmospheric CO_2 content in the last two centuries up to present. The final section concludes.

2 Growth models

2.1 Generalization of Exponential Growth

The benchmark for population growth is the Malthus model, which postulates that population growth is proportional to the population itself, capturing the simple idea that the number of children is proportional to the number of parents:

$$\frac{dp}{dt} = r \cdot p(t) \ . \tag{2}$$

The solution of equation (2) is the exponential function

$$p(t) = a' \exp(r \cdot t) + c' \ . \tag{3}$$

Historically, equation (2) has been improved by [26, 27] into the logistic equation, to account for the competition for scarce resources between individuals. This competition can be embodied into the quadratic term $-r[p(t)]^2/K$, where K is the carrying capacity. This negative feedback of the population on the growth rate $r \to r(1 - p(t))/K$ leads to a cross over from the exponential growth for $p(t) \ll K$ to a saturation of the population at long times, which asymptotically converges to K. Verhulst thought that Malthus was wrong (and therefore over-pessimistic when comparing human growth with food resources) not to take into account the negative feedbacks embodied in the quadratic term $-r[p(t)]^2/K$, that would lead naturally to an equilibrium.

But, the human population at the time of Verhulst and until around 1960 has followed neither his specification, nor the Malthusian exponential growth. As reviewed by [12] and references therein, the human population has grown faster than exponential, with the growth rate r growing itself.

The simplest generalization of equation (2) that accounts for this observation assumes that the growth rate r becomes $r \cdot [p/p_0]^\delta$, where $\delta > 0$ and

p_0 is some reference population. The positivity of δ captures the positive feedback of population on the growth rate: the larger the population, the larger the growth rate! Equation (2) then transforms into

$$\frac{dp}{dt} = R \cdot p(t)^{1+\delta} dt \ , \tag{4}$$

where $R = r/p_0^\delta$. The solution of equation (4) reads

$$p(t) \propto (t_c - t)^{-1/\delta} \qquad \text{if } \delta > 0 \ . \tag{5}$$

Here, the critical time t_c at which the solution diverges is determined from the parameters of equation (4) and the initial population. For $\delta = 0$, we recover the exponential solution (3), seen as the limit of a finite-time-singularity (FTS) power law with exponent tending to zero. The singular solution (5) was first discussed by von Foerster et al. [28] (see [25] for assessments of the relative merits of the "natural science" versus the "demographic" approach, [14] for an economic underpinning that we explore later, and [12,13] for extensive generalizations). In ecology, the positive correlation between population density and the per capita population growth rate at the origin of the FTS behavior (5) is known as the Allee effect, see for instance [24]. More generally, Allee discovered the existence of an often present positive relationship between some component of individual fitness and either numbers or density of conspecifics. The Allee effect is usually used to refer to the self-reinforcing feedbacks that promote accelerate extinction of species, that can be modeled by finite-time crossing of zero, see [31]. Goriely provides a rigorous mathematical framework [9] with a generalized version of equation (4), where the right hand side is replaced by an arbitrary polynomial of $p(t)$.

The use of the mathematics of FTS to describe and diagnose changes of regime is not new. For instance, we refer to [12, 22] for population dynamics and financial markets, [19] for applications to engineering failures and earthquakes, [21, 23] for a large variety of systems, [5] for climate systems, and [2, 6, 20] for environmental systems. These authors applied the concept of dynamical phase transitions and FTS to different systems exhibiting a bifurcation, crisis, catastrophe or tipping point, by showing how specific signatures can be used for advance warnings.

One can generalize (4) to take into account positive feedbacks of the growth rate $d \ln p/dt$ on its rate of change $d^2 \ln p/dt^2$ (see [11]), to arrive at solutions that exhibit FTS not in the variable $p(t)$, but in its derivative dp/dt. We will thus use the slightly more general expression encompassing these cases:

$$p_{\text{power}}(t) = a(t_c - t)^{-1/\delta} + c \ . \tag{6}$$

A FTS in dp/dt and not in $p(t)$ corresponds to $-\infty < \delta < -1$ such that $0 < -1/\delta < 1$, together with $a < 0$ for an increase up to the value $p_{\text{power}}(t_c) = c$. Here, the meaning of the exponent δ is different from its use in equation (4).

We shall use the *exponential model* (3) and the *power law model* (6), as our two competing hypotheses. The essential difference between the *exponential model* and the power law model is that the former is defined for all times, while the later is valid only up to a finite time, the critical time t_c beyond which the solution ceases to exist. The singular behavior at t_c is not meant to predict a genuine divergence but only, as already stressed, that the system is exhibiting a transition to a qualitatively new regime.

2.2 Properties distinguishing the exponential and the power law model

Heated discussions among demographers greeted the publication of the paper [28] concerning the singular solution (5): the demographers criticized the use of mathematical models such as (4) as perhaps the clearest illustration of how bad use of mathematics may yield senseless results; actually, what the demographers missed was that the FTS should not be taken at face value, but as the signature of a transition to a new regime. Singularities do not exist in natural and social systems, but the singularities of our approximate mathematical models are usually very good diagnostic of the change of regimes that occur in these systems. The perhaps clearest examples are the phase transitions between different states of matter (solid-liquid-gas-plasma, magnetized to non-magnetized, and so on) that statistical physics describes so well with its classification involving the nature of the singularity exhibited by the free energy of the system [8].

As t approaches t_c from below, two regimes can be observed for the power law model:

$\delta < 0$: $(t_c - t)^{-1/\delta}$ goes to zero for $t \to t_c$ and $p_{\text{power}}(t) \to c$.

$\delta > 0$: $(t_c - t)^{-1/\delta}$ goes to infinity for $t \to t_c$ and $p_{\text{power}}(t) \to \text{sign}(a) \cdot \infty$.

Figure 6 in the SI illustrates the qualitatively different behaviors allowing one to distinguish between the linear growth model ($dp(t)/dt \sim t$), the *exponential model* (3) and the *power law model* (6), in different standard plot representations.

2.3 Faster-than-exponential growth by feedbacks between macroeconomic variables

Up to now, we have postulated the form (4) to capture the possible existence of a positive feedback of population on the population growth rate. Such a simplified ansatz leaves two issues unresolved. First, the positive feedback of population on growth rate may not be direct, but mediated by other variables via indirect mechanisms. Second, the consequences on the dynamics of carbon dioxide emissions are not clear. We thus address these two issues using an economic framework developed by Kremer [14], following the approach of Johansen and Sornette [12]. The following derivation is not intended to represent a faithful economic growth model that we would like to promote, but is offered to illustrate the importance of indirect mechanisms in growth processes. In particular, we would like to stress the fact that faster-than-exponential growth is a robust outcome of multi-dimensional loop processes. Even when each feedback process individually leads to an exponential or even a subdued sub-exponential growth, the overall dynamics can be super-exponential.

In economics, population $p(t)$ translates into labor force $L(t)$, which is assumed to be proportional to $p(t)$. In addition to population represented by the labor force, we consider the effect of technology level $A(t)$ and of the amount $K(t)$ of available capital. In the presence of labor and capital, with a given technology level, the economy is going to produce an output $Y(t)$, for instance proxied by GDP. In the macroeconomics of endogenous growth [18], it is common to use the Cobb-Douglas equation (originally developed by [4] and extensively discussed in [18]) to relate the total output to labor, capital and technology as follows[3]:

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \text{ , with } \ 0 < \alpha < 1 \ . \tag{7}$$

Furthermore, we use the assumption by Solow that a constant fraction s of the economy goes to savings , i.e. capital grows according to

$$\frac{dK}{dt} = sY(t) \ . \tag{8}$$

Following [14], we assume that, as already mentioned, labor is proportional to capital

$$K(t) \sim L(t) \ . \tag{9}$$

We further assume that technology change is depending on capital, labor and current level of technology according to

$$\frac{dA}{dt} = dK(t)^{\eta} \times L(t)^{\gamma} \times A(t)^{\theta} \ , \tag{10}$$

where the exponents η, γ and θ are all positive, expressing a positive feedback effect of each of the variables on the growth of technology. Putting together all these ingredients, we can rewrite the Kremer (9) and Solow (8) equations

[3] A' in the IPAT equations stands for gross world product per capita, whereas in the Cobb-Douglas equation A stands for technology. Further, the IPAT equation uses T instead of A to denote technology. Similar, the macro-economists refer to L as labor, whereas P in the IPAT equality stands for population. We will not distinguish between labor L and population P and use the terms interchangeably.

as a system of two coupled ordinary differential equations:

$$\frac{dA}{dt} = eL(t)^{\eta+\gamma} \times A(t)^{\theta}, \tag{11}$$

$$\frac{dL}{dt} = fL(t) \times A(t)^{1-\alpha} \ . \tag{12}$$

Equation (12) basically states that labor (and thus population) is growing exponentially, holding technology constant. In other words, the growth rate of population is controlled by a nonlinear function of technology. Here, this nonlinear function is a power law with exponent $0 < 1 - \alpha < 1$, which embodies the benefits that technology brings in decreasing death rates, for instance via improvement in health care. Invoking this mechanism is standard in demographic research.

We look for solutions exhibiting a FTS of the form

$$A(t) = A_0(t_c - t)^{-1/\mu} \ , \tag{13}$$

$$L(t) = L_0(t_c - t)^{-1/\kappa} \ . \tag{14}$$

Note that the critical time t_c of the singularity, if it exists, is necessarily the same for both variables, as seen from inspection of the two coupled equations (11,12). Inserting this ansatz in equations (11,12), we obtain a system of differential equations for the unknown exponents μ and κ, whose solutions read

$$\mu = 1 - \alpha \ , \tag{15}$$

$$\kappa = \frac{\eta + \gamma}{2 - \theta - \alpha}(1 - \alpha) \ . \tag{16}$$

The condition for the solutions (13,14) to hold is that μ and κ be strictly positive. This implies $0 < \alpha < 1$ and $\alpha < 2 - \theta$. If $\theta \leq 1$, then the conditions are always satisfied in the regime where the Cobb-Douglas equation holds. The case $\theta \leq 1$ is particularly interesting because it corresponds to a sub-exponential growth of technology, for a fixed labor force. In other words, for a fixed population level, equation (11) gives a long-time growth of the form $A(t) \sim t^{\frac{1}{1-\theta}}$, which is sub-exponential (slower than exponential) for $\theta < 1$ and exactly exponential for $\theta = 1$. It is the coupling between a sub-exponential growth of $A(t)$ and an exponential growth of population $L(t)$ mediated by nonlinear feedback loops that create the super-exponential finite-time singularity. This behavior underlies the possible traps of single variable analysis.

These results can be translated into a prediction of carbon dioxide emission via the following simple assumption. Assuming that carbon dioxide emissions are proportional to production divided by some power of technology ξ, we have

$$\frac{d\mathrm{CO}_2}{dt} = \frac{Y(t)}{A(t)^{\xi}} = h(t_c - t)^{-1/\varphi}, \tag{17}$$

where $\varphi = (1/\kappa - \xi/\mu + 1)^{-1}$ (see SI for details of the derivation) and CO_2 stands for the total carbon dioxyde content in the atmosphere. The introduction of a non-zero exponent ξ accounts for the common observation that more developed countries tend to have a lower footprint and smaller carbon emissions per unit of output, due to the progressive adoption of more efficient technologies and the increasing importance of a clean environment in the utility functions of consumers.

Let us thus stress the main result of this exercise. We have $\frac{dA}{dt} \sim A(t)^\theta$ at fixed labor with $\theta < 1$ and $\frac{dL}{dt} \sim L(t)$ at fixed technology. Thus, there is no way to get a faster-than-exponential growth in any of these two variables alone. However, when coupling them via the feedback of labor on technology and that of technology on labor, the FTS power law solutions (13,14) emerge. Hence, a finite-time singularity can be created from the interplay of several growing variables resulting in a non-trivial behavior: the interplay between different quantities may produce an "explosion" in the population even though the individual dynamics do not!

Of course, infinities do not exist on a finite Earth! These singularities should not be interpreted as the prediction of real "blow-ups". They can be however faithful description of the transient dynamics up to a neighborhood of the predicted critical time t_c. Around t_c, new mechanisms kick in and produce a *change of regime*.

To illustrate the above point, let us go through a detailed scenario where the individual processes stay finite in finite time, but the combination via feedback can lead to finite time singularities. Consider the following parameters

$\alpha = \frac{1}{4}$: as in the seminal paper [4].

$\theta = 1$: Linear feedback from technology A on itself. Holding all other factors constant, technology will grow exponentially (see equation (10)).

$\eta + \gamma = 1$: The simplest possible, non-trivial, assumption.

With these numbers, we obtain the two exponents $\mu = 3/4$ and $\kappa = 1$ for the equations (13) and (14), respectively, and the value $1/\varphi = 5/3$ for the rate of carbon dioxide emission given by equation (17), assuming carbon dioxide emission per capita technology is as efficient as general technology A, i.e. $\alpha = \xi = 1/4$. Although, we have only assumed exponential growth of all individual factors, carbon dioxide emission is predicted in this example to grow faster than exponential, leading to a mathematical FTS which is the signature of a non-sustainable regime towards a new behavior (see Figure 8).

Even less stringent conditions for a FTS to occur are needed when the description of the dynamics of the system in terms of two coupled equations (11,12) is augmented to take into account the dynamics of additional coupled variables, leading to systems of three or four coupled equations. Such additional positive feedback loops include nonlinear lagged dependencies of capital on labor (thus extending Kremer's simplifying assumption (9)).

3 Empirical tests on human population and atmospheric carbon dioxide content

3.1 Population

Figure 7 shows that the growth rate of the World population was a strongly increasing function of time till the late 1950s. A sharp decrease of the growth rate occurred, then followed by a resumed acceleration till its peak in 1964, from which a slow decrease can be observed.

The first regime till about 1960 is incompatible with the exponential model, which corresponds to a constant growth rate. Figure 1 shows that, over the time period 1850 to 1965, the exponential model is inferior to the FTS power law model. Using model (4), we estimate that the growth exponent δ is approximately equal to 2, that is, even larger than the value 1 estimated by [28]: clearly, population growth over this time period was faster than exponential and the FTS power law model accounts parsimoniously for the data.

Figure 2 shows that, over the time period from 1970 to 2008, the exponential model (3) and the FTS power law model (6) are indistinguishable. By Occam's razor, the exponential model with an approximately constant growth rate is then preferred.

3.2 Carbon Dioxide content in the atmosphere

Figure 9 in the SI plots the carbon dioxide content in the atmosphere since 1000 CE. The dramatic acceleration due to anthropogenic forcing since the 1800s is clearly observed.

We calibrate the *exponential model* (3) and the *power law model* (6) separately to two time periods: (i) from 1850 to 1954 (Figure 3), for which the data originates from ice drill cores and (ii) from 1959 to 2009 (Figure 4), for which the data originates from air samples. The quality of the fits by the two models, as quantified by the sum of squared errors between theory and data, is practically equivalent. Therefore, we cannot reject the hypothesis that the exponential model is sufficient to explain the data for each time window *separately*.

However, the growth rate r calibrated with the exponential model (see equation (3) has more than doubled from the first period 1850 – 1954 ($r = 0.0066$) to the second period 1959 – 2009 ($r = 0.016$). While being not fully warranted given the heterogeneity of the data sources, we have fitted the two models to the whole period from 1850 to 2009. We find that the FTS power law is the clear winner (see Figure 10) which, together with the more than doubling of the growth rate r from the first to the second time intervals, suggests the existence indeed of a faster-than-exponential growth of the atmospheric content of carbon dioxide.

We now attempt to be more precise on the nature and evolution of the faster-than-exponential growth by estimating the exponent δ of equations (6) applied to the time series of carbon dioxide atmospheric content. We use the monthly data from the Mauna Loa site, as it is considered to be one of the most reliable. Before calibrating equation (6) to various time intervals $[t_1, t_2]$, we smooth the data by using a Gaussian kernel with a width of 10 years. Then, we estimate δ, with t_1 being scanned from 1958 to 2006 and t_2 being scanned from 1960 to 2009 as shown in Figure 5.

Two main results are obtained. First, the exponent δ is found almost always larger than or equal to 1, implying a growth at least as fast as exponential and often significantly faster. Second, one can observe a systematic trend. For time intervals starting earlier (i.e., for t_1 in the late 1950s and in the 1960s), the exponent δ tends to be closer to 1, while for larger t_1, δ is significantly larger than 1. This leads to the conclusion that the carbon dioxide content in the Earth atmosphere is growing at least exponential and probably faster-than-exponentially, with no sign of abating. The latest time intervals are characterized by the largest exponents δ's, significantly above the lower bound 0 that would correspond to an exponential growth. The content of carbon dioxide in the atmosphere is accelerating super-exponentially.

3.3 Compatibility between exponential population growth and super-exponential CO2 emissions

The previous empirical evidence suggests that the human population on the Earth is growing now just exponentially, while there is suggestive evidence that the content of carbon dioxide in the atmosphere is accelerating super-exponentially. How are these two different behaviors be compatible with the solutions (13,14) for $A(t)$ and $L(t)$ of equations (11,12)?

We consider two possible explanations. The first one would argue that until the 1960s both population and atmospheric carbon dioxide content were super-exponentially accelerating in accordance with expressions (13,14). Then, the slowing down from super-exponential to just exponential growth of the human population could be interpreted as a finite-size effect that is starting to be felt for this variable only, as physical limits are more stringent for the human carrying capacity and the response of human birth and death rates to policies than they are for carbon dioxide emissions.

The second explanation is that the two different behaviors of $A(t)$ and $L(t)$ may be resolved within the mathematical structure developed in equations (13) and (14). Indeed, let us assume that the growth of the human population is following solution (14), but with a small value of the exponent κ. For all practical purpose, a FTS power law with a small exponent is indistinguishable from an exponential growth over a finite time interval. This interpretation is reasonable in so far that human population growth has been unambiguously super-exponential until the 1960s, and it is only recently that this growth has somewhat abated. It is thus quite possible that it is still super-exponential but to a degree that is not sufficiently strong to be distinguishable from a pure exponential, as shown in the analysis of Figures 1 and 2.

Let us now turn to the dynamics of CO_2 content. The conditions for a super-exponential growth of the content of carbon dioxide in the atmosphere are compounded by many complex processes involving, in addition to the emissions, the sequestrations of CO_2 by, and dynamics of, the ocean and biosphere. As a rough rule of thumb, we assume that the total content of carbon dioxide in the atmosphere at time t is simply proportional to (but likely less than) the cumulative release of CO_2 until time t. In other words, CO_2 content is estimated as a finite fraction of the solution of equation (17). Under these assumptions, in order for CO_2 content to exhibit a FTS power law behavior, it is necessary and sufficient that the exponent $1/\varphi$ in (17) be larger than 1. Indeed, by integration, $CO_2(t)$ remains of the same form $(t_c - t)^{-1/\delta}$, with $1/\delta = 1/\varphi - 1 > 0$, where δ is defined as in equation (6). This condition translates into the condition $\xi < \mu/\kappa$. As we have assumed that κ is small, corresponding to the closeness of the population dynamics to an exponential growth, this condition does not provide a strong constraint for ξ: CO_2 content can exhibit an (accelerated) FTS dynamics even if ξ is large, corresponding to a more efficient economy. If $1 - \alpha$ is close to 0, corresponding to output mainly controlled by availability of capital, then ξ should be small. Small values of ξ correspond to the situation in which, taken globally over the whole Earth, the technological advances have not yet significantly abated carbon emission per unit of output. This statement may appear shocking and counter-factual for developed countries. But, at the scale of the whole planet, one can observe that improvement in carbon emissions (i.e., decrease per unit of output) in the developed countries are counteracted by the increases of carbon emissions in some major developing countries, such as China, India and Brazil, which use carbon emission inefficient technologies (for instance heavily based on coal burning). In summary, we find a very robust FTS behavior for CO_2 over a broad and realistic range of parameters, which makes it difficult to constrain the impact of the advance of technology on production efficiency.

4 Conclusion

We have analyzed the growth of atmospheric carbon dioxide and of what constitutes arguably its most important underlying driving variable, namely human population. Our empirical calibrations suggest that human population has decelerated from its previous super-exponential growth until 1960 to "just" an exponential growth. As for atmospheric CO_2 content, we find that it is at least exponentially increasing and more probably exhibiting an accelerating growth rate, consistent with a FTS (finite-time singular) power law regime.

We have proposed a simple framework to think about these dynamics, based on endogenous economic growth theory. We showed that the positive feedback loops between several variables, such as population, technology and capital can give rise to the observed FTS behavior, notwithstanding the fact that the dynamics of each variable would be stable or at most exponential, conditional on the stationarity of the other variables. It is the joint growth of the coupled variables that may give rise to the enormous acceleration characterized by the FTS behavior, both in the equation and, we present suggestive evidence, in the carbon dioxide content in the atmosphere.

Overall, the evidence presented here does not augur well for the future.

- The human population is still growing at an exponential rate and there is no sign in the data that the growth rate is decreasing. Many argue that economic developments and education of women will lead to a decrease growth rate and an eventual stabilization of human population. This is not yet observed in the population dynamics, when integrated worldwide. Let us hope that the stabilization of the human population will occur endogenously by self-regulation, rather than by more stringent finite carrying capacity constraints that can be expected to lead to severe strains on a significant fraction of the population.

- Notwithstanding a lot of discussions, international meetings, prevalence in the media, atmospheric CO_2 content growth continues unabated with a clear faster-than-exponential behavior. On the face of this evidence using data until 2009, stabilizing atmospheric carbon dioxide emissions at levels reached in 1990 for instance seems very ambitious, if not utterly unrealistic. We are not pessimistic. We think that only evidence-based decision making can lead to progress. The present evidence gives some measure of the enormous challenges to control our CO_2 emissions to acceptable levels.

5 Data

Population data was obtained from the website of the United Nations (`http://www.un.org/esa/population/publications/sixbillion/sixbilpart1.pdf`) and the website of U.S. Census Bureau (`http://www.census.gov/ipc/www/idb/worldpop.php`).

Carbon dioxide data was collected from different sources: from the Carbon Dioxide Information Analysis Center (CDIAC) (`http://cdiac.esd.`

`ornl.gov/ftp/trends/co2/siple2.013`, `http://cdiac.ornl.gov/ftp/trends/co2/lawdome.combined.dat`), the National Oceanic and Atmospheric Administration (NOAA) (`ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_annmean_mlo.txt`, `ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/maud/edml-co2-2005.txt`, `ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_mm_mlo.txt`) and from [1][4].

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A Discussion of exponential growth / FTS power-law

Depending on the scale of the abscissa and the ordinate, exponential growth and FTS power-law growth can look very different (see also Figure 6):

- (a): the linear-linear plot shows the dual property of the FTS power law function, which is to both grow initially slower than the two other models, and then to catch up explosively.
- (b): in this linear-log plot, by construction, the exponential function is a straight-line, thus a linear dependence in this representation qualifies an exponential growth. The linear model is concave (slower growth) and the power law FTS model is convex (faster growth).
- (c): the log-log plot would qualify a power law t^{β} as a straight line whose slope is the exponent β. Hence the linear function is also linear in this representation with slope 1. Both the exponential and FTS power law model exhibit an upward convex shape. It is important not to confuse a power law and a FTS power law: the former is proportional to a power of t and thus exists for all times, while the later is proportional to a power of $t_c - t$ and is only defined for $t < t_c$.
- (d): in this log-log plot in the variable $t_c - t$, by construction, the FTS power law is qualified by a straight line behavior, with a slope equal to the exponent $-1/\delta$. Both linear and exponential models are associated with concave curves, characterizing a slower growth in the vicinity of t_c. Note that time t increases to the left.

B Exact Solution of the ODE system

This appendix provides the exact derivation of the system of equations (11) and (12), thus justifying the ansatz (13) and (14) used.

First, we combine equations (11) and (12) into a single equation:

$$\frac{dA}{dt}L(t)^{-\eta-\gamma}A(t)^{-\theta} - \frac{dL}{dt}L(t)^{-1}A(t)^{-1+\alpha} = 0 \ . \tag{18}$$

Without loss of generality, we can set $e = f = 1$ by defining appropriately the units of A and L. Separating the variables and integrating lead to

$$\frac{1}{2-\alpha-\theta}A(t)^{2-\alpha-\theta} - \frac{1}{\eta+\gamma}L(t)^{\eta+\gamma} = c' \ . \tag{19}$$

Looking for the large time asymptotic regime for which $L(T)$ and $A(t)$ (which are assumed to be monotonously increasing) become much larger that the constant c', we can solve for $A(t)$ and $L(t)$ as follows.

- Hence,

$$L(t) = \left[\frac{1}{2-\alpha-\theta}A(t)^{2-\alpha-\theta}(\eta+\gamma)\right]^{1/(\eta+\gamma)} \tag{20}$$

$$= c_2 A(t)^{\frac{2-\alpha-\theta}{\eta+\gamma}} \ . \tag{21}$$

Plug-in this into equation (11) leads to

$$\frac{dA}{dt} = c_2 A(t)^{2-\alpha} \ . \tag{22}$$

By separating variables and subsequent integration, we get:

$$A(t)^{\alpha-2}dA = c_2 dt \ , \tag{23}$$

$$\frac{1}{\alpha-1}A(t)^{\alpha-1} = c_2 t + c_2' \ , \tag{24}$$

$$A(t) = \left[(1-\alpha)c_2(\frac{-c_2'}{c_2} - t)\right]^{-1/(1-\alpha)} \tag{25}$$

$$\Leftrightarrow A(t) = A_0(t_c - t)^{-1/\mu} \ , \tag{26}$$

with $\mu = 1-\alpha$.

- Similar, we find the solution for $L(t)$:

$$A(t) = \left[\frac{1}{\eta+\gamma}L(t)^{\eta+\gamma}\right]^{1/(2-\alpha-\theta)} \tag{27}$$

$$= c_3 L(t)^{\frac{\eta+\gamma}{2-\alpha-\theta}} \ . \tag{28}$$

Plug-in this into equation (12) leads to

$$\frac{dL}{dt} = c_3' L(t)^{\frac{(\eta+\gamma)(1-\alpha)}{2-\alpha-\theta}+1} \tag{29}$$

$$=: c_3' L(t)^{\kappa+1} \quad \text{where } \kappa := \frac{(\eta+\gamma)(1-\alpha)}{2-\alpha-\theta} \ . \tag{30}$$

As before, we separate variables and integrate

$$L(t)^{-\kappa-1}dL = c_3' dt \ , \tag{31}$$

$$\frac{1}{-\kappa}L(t)^{-\kappa} = c_3' t + c_3'' \ , \tag{32}$$

$$L(t) = \left[\kappa c_3'(\frac{-c_3''}{c_3'} - t)\right]^{-1/\kappa} \tag{33}$$

$$\Leftrightarrow L(t) = L_0(t_c - t)^{-1/\kappa} \ , \tag{34}$$

with $\kappa = \frac{\eta+\gamma}{2-\alpha-\theta}(1-\alpha)$.

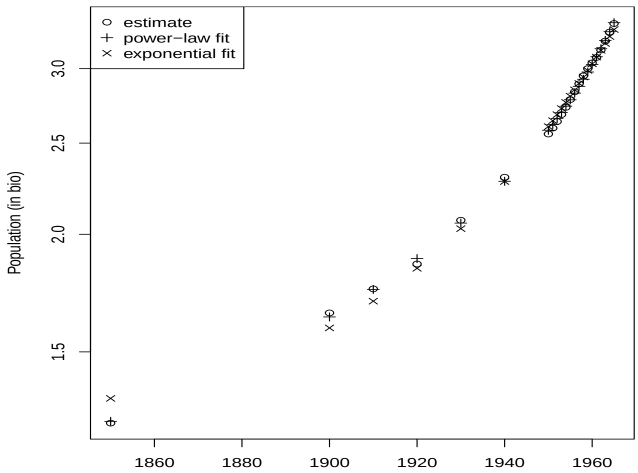

Figure 1: Population data represented by the empty circles (where "estimate" refers to the empirical estimation of the population) fitted over the time window from 1850 – 1965 by the FTS power-law (6) and the exponential model (3). The fitted parameters are $\delta = 2$ and $t_c = 1988$ for the power-law and $r = 0.028$ for the exponential fit.

Of course, the solution for $L(t)$ could be directly obtained using (21) and (26), and reciprocally.

For a general mathematical rigorous theory of ordinary differential equations exhibiting finite-time singular behaviors, see [9].

C Calculation of the exponent φ

Let us give some intermediate steps towards the solution of equation (17).

$$
\begin{aligned}
\frac{Y(t)}{A(t)^\xi} &=^{(7)} \frac{K(t)^\alpha (A(t)L(t))^{1-\alpha}}{A(t)^\xi} && (35)\\
&=^{(9)} L(t)A(t)^{1-\alpha-\xi} && (36)\\
&=^{(13,14)} L_0(t_c-t)^{-1/\kappa}\left[A_0(t_c-t)^{-1/\mu}\right]^{1-\alpha-\xi} && (37)\\
&= L_0A_0(t_c-t)^{-1/\kappa-(1-\alpha-\xi)/\mu} && (38)\\
&=^{!} C_0(t_c-t)^{-1/\varphi}\ . && (39)
\end{aligned}
$$

Hence,

$$
\varphi = \frac{1}{1/\kappa - \xi/\mu + 1}\ . \tag{40}
$$

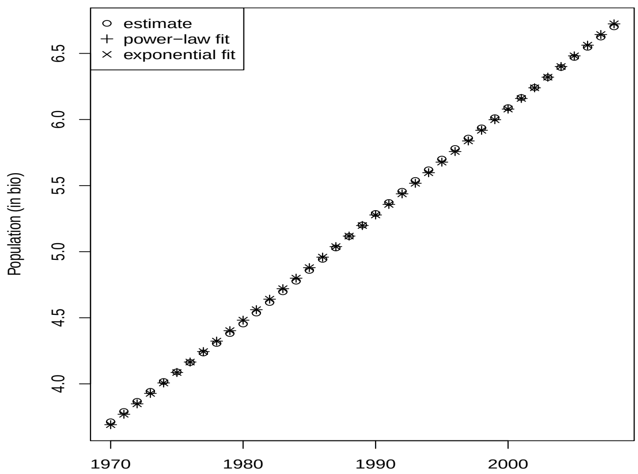

Figure 2: Population data fitted over the time window from 1970 – 2008 by the FTS power-law (6) and the exponential model (3). The fitted parameters are $\delta = 3.5$ and $t_c = 3939$ for the power-law and $r = 0.00067$ for the exponential fit.

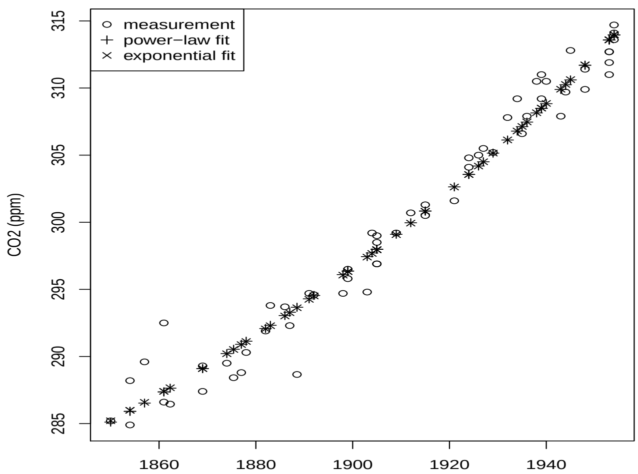

Figure 3: Carbon dioxide data fitted over the time window from 1850 – 1954 by the FTS power-law (6) and the exponential model (3). The fitted parameters are $\delta = 0.65$ and $t_c = 2304$ for the power-law and $r = 0.0066$ for the exponential fit. The two fits are almost undistinguishable and their goodness-of-fit is essentially the same.

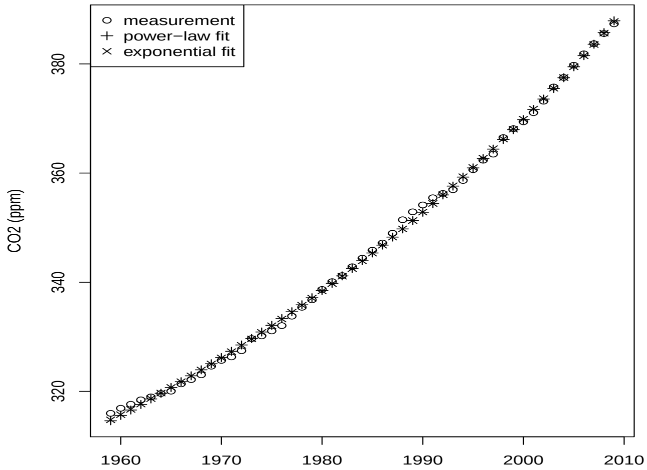

Figure 4: Carbon dioxide data fitted over the time window from 1959 – 2009 by the FTS power-law (6) and the exponential model (3). The fitted parameters are $\delta = 0.73$ and $t_c = 2132$ for the power-law and $r = 0.016$ for the exponential fit. The two fits are almost undistinguishable and their goodness-of-fit is essentially the same.

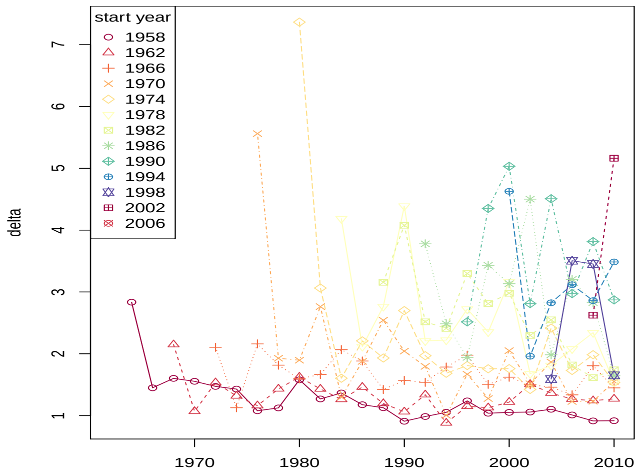

Figure 5: Estimates of the exponent δ of equation (6) on the monthly Mauna Loa carbon dioxide data obtained from air measurements in different intervals $[t_1, t_2]$. Each line corresponds to a specific start time t_1, as shown in the legend. The ending point t_2 is the variable on the abscissa.

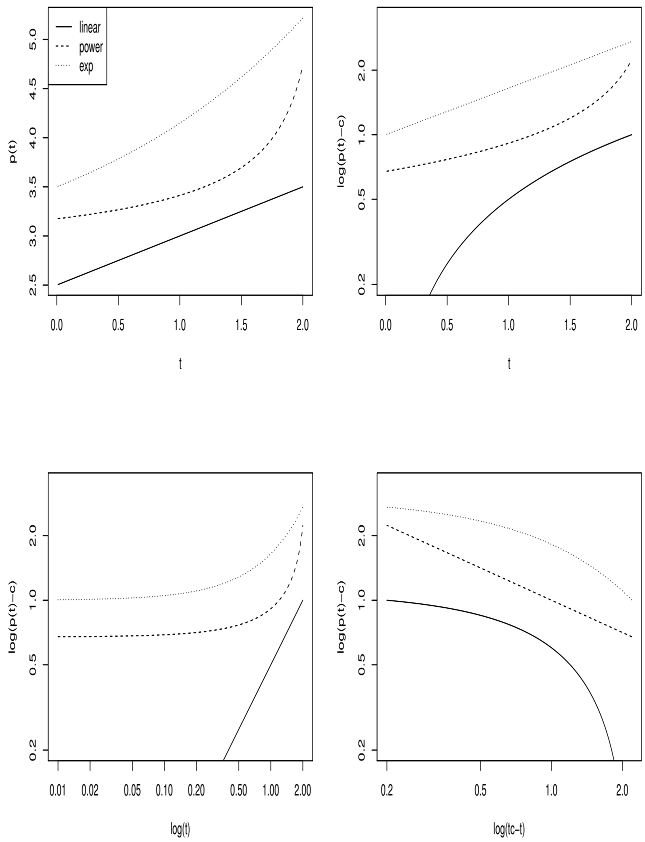

Figure 6: Illustration of the qualitatively different behaviors of the *exponential model* (3), the *power law model* (5) and a *linear model*, in different standard plot representations. For each of the four plots, the linear function $0.5t + 3.25$ is compared with the exponential function $1e^{0.5t} + 2.5$ and with the power law $1(2.2 - t)^{-0.5} + 2.5$. (a) is linear-linear, (b) is linear-log, (c) is log-log and (d) is log-log referenced to the singularity. The constant c is set equal 2.5. The relative vertical positions of the three curves are arbitrarily chosen (from the above values) for the sake of a clear visualization.

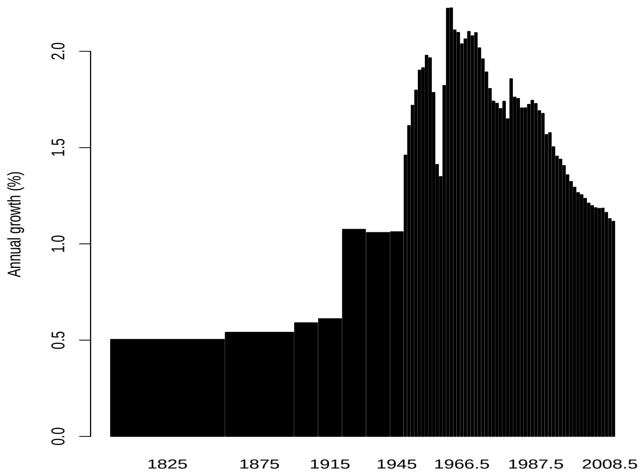

Figure 7: Annualized world population growth rate from year 1800 – 2010.

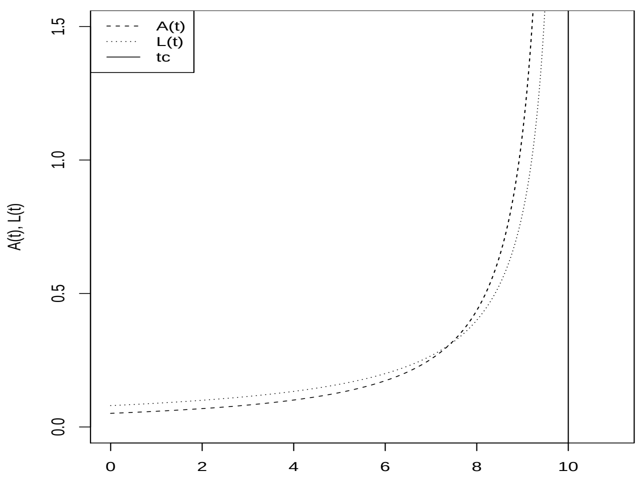

Figure 8: Numerical solution of equations (11) and (12) with $\alpha = \frac{1}{4}$, $\eta + \gamma = 1$, $\theta = 1$ and $t_c = 10$. The initial conditions are $A(0) = 1.1$ and $L(0) = 0.8$. We assume without loss of generality $e = f = 1$, as these coefficients can be absorbed in the units of A and L respectively. $L(t)$ and $A(t)$ grow super-exponentially towards a singularity occurring at the same time as a result of their coupling. The logarithms of $A(t)$ and $L(t)$ are plotted as a function of (linear time). The upward curvatures and approaches to the singular vertical asymptote exemplify the super-exponential growth.

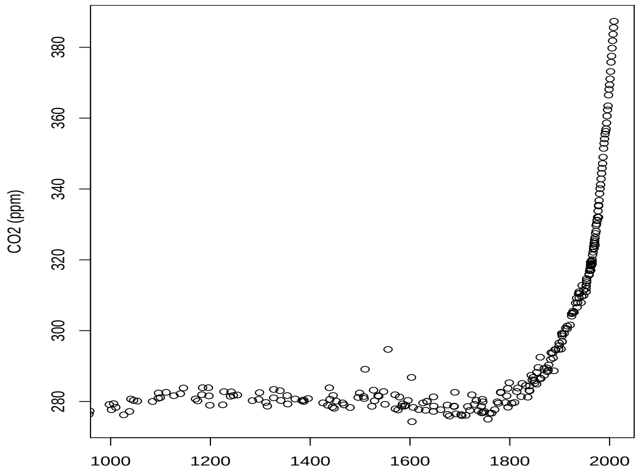

Figure 9: Atmospheric carbon dioxide since 1000 CE to present. The data shown combines ice core and air measurements from different sources. See data section for more details.

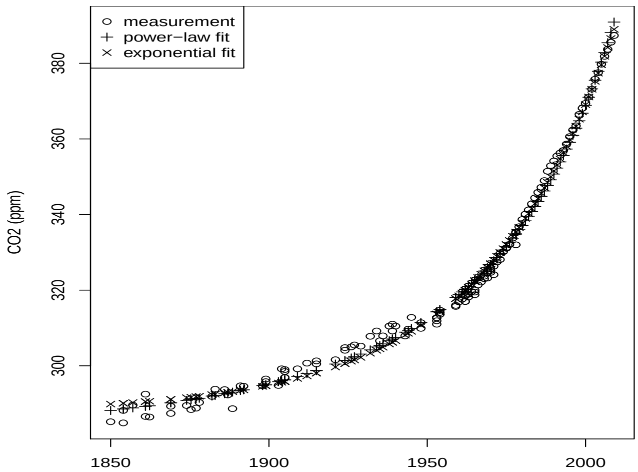

Figure 10: Carbon dioxide data fitted over the time window from 1850 – 2009 by the FTS power-law (6) and the exponential model (3). The fitted parameters are $\delta = 0.33$ and $t_c = 2129$ for the power-law and $r = 0.024$ for the exponential fit. The ratio of squared errors between the power-law and the exponential-fit is 0.88.